

Note on K-stability of pairs

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Abstract

We prove that a pair (X, D) with X Fano and D a smooth anti-canonical divisor is K-unstable for negative angles, and K-semistable for zero angle.

1 Introduction

Let X be a Fano manifold. It was first proposed by Yau([19]) that finding Kähler-Einstein metrics on X should be related to certain algebro-geometric stability. In [17], the notion of K-stability was introduced by Tian. This has been conjectured to be equivalent to the existence of a Kähler-Einstein metric. One direction is essentially known, in a wider context of constant scalar curvature Kähler metrics([3]). Namely, it is proved by Donaldson([4]) that the existence of a constant scalar curvature metric implies K-semistability. This was later strengthened by Stoppa([15]) to K-stability in the absence of continuous automorphism group, and by Mabuchi([9]) to K-polystability in general.

Recently in [6](see also, [16], [7]) the K-stability has been defined for a pair (X, D) , where X is a Fano manifold and D is a smooth anti-canonical divisor. The definition involves a parameter $\beta \in \mathbb{R}$. At least when $\beta \in (0, 1]$, the K-stability of a pair (X, D) with parameter β is conjectured to be equivalent to the existence of a Kähler-Einstein metric on X with cone singularities of angle $2\pi\beta$ transverse to D . This generalization grew out of a new continuity method for dealing with the other direction of the above conjecture, as outlined in [5]. Note heuristically the case $\beta = 0$ corresponds to a complete Ricci flat metric on the complement $X \setminus D$. By the work of Tian-Yau([18]) such a metric always exists if D is smooth. In this short article we prove the following theorem, which may be viewed as an algebraic counterpart of the differential geometric result of Tian-Yau.

Theorem 1.1. *Any pair (X, D) is strictly K-semistable with respect to angle $\beta = 0$, and K-unstable with respect to angle $\beta < 0$.*

This has an immediate consequence

Corollary 1.2. *If X is K -stable(semi-stable), then for any smooth anti-canonical divisor D , the pair (X, D) is K -stable(semi-stable) with respect to angle $\beta \in (0, 1]$.*

The corollary provides evidence to the picture described in [5] that a smooth Kähler-Einstein metric on X should come from a complete Calabi-Yau metric on $X \setminus D$ by increasing the angle from 0 to 2π . The relevant definitions will be given in the next section. The strategy to prove K -unstability for negative angle is by studying a particular test configuration, namely the deformation to the normal cone of D . To deal with the zero angle case we shall construct “approximately balanced” embeddings using the Calabi-Yau metric on D . In [11], Odaka proved that a Calabi-Yau manifold is K -stable, by a purely algebro-geometric approach. It is very likely that his method can give an alternative proof of the above theorem, but the one we take seems to be more quantitative.

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2 K -stability for pairs

We first recall the definition of K -stability.

Definition 2.1. Let (X, L) be a polarized manifold. A *test configuration* for (X, L) is a \mathbb{C}^* equivariant flat family $(\mathcal{X}, \mathcal{L}) \rightarrow \mathbb{C}$ such that $(\mathcal{X}_1, \mathcal{L}_1)$ is isomorphic to (X, L) . $(\mathcal{X}, \mathcal{L})$ is called *trivial* if it is isomorphic to the product $(X, L) \times \mathbb{C}$ with the trivial action on (X, L) and the standard action on \mathbb{C} .

Suppose D is a smooth divisor in X , then any test configuration $(\mathcal{X}, \mathcal{L})$ induces a test configuration $(\mathcal{D}, \mathcal{L})$ by simply taking the flat limit of the \mathbb{C}^* orbit of D in \mathcal{X}_1 . We call $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ a *test configuration for (X, D, L)* . Given any test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ for (X, D, L) , we denote by A_k and \tilde{A}_k the infinitesimal generators for the \mathbb{C}^* action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ and $H^0(\mathcal{D}_0, \mathcal{L}_0^k)$ respectively. By general theory for k large enough we have the following expansions

$$\begin{aligned} d_k &:= h^0(\mathcal{X}_0, \mathcal{L}_0^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \\ w_k &:= \text{tr}(A_k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}), \\ \tilde{d}_k &:= h^0(\mathcal{D}_0, \mathcal{L}_0^k) = \tilde{a}_0 k^{n-1} + \tilde{a}_1 k^{n-2} + O(k^{n-3}), \\ \tilde{w}_k &:= \text{tr}(\tilde{A}_k) = \tilde{b}_0 k^n + \tilde{b}_1 k^{n-1} + O(k^{n-2}). \end{aligned}$$

Definition 2.2. For any real number β , the *Futaki invariant* of a test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ with respect to angle β is

$$Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1 - \beta)(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0} b_0).$$

When $\beta = 1$ we get the usual Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$

$$Fut(\mathcal{X}, \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0}.$$

Definition 2.3. A polarized manifold (X, L) is called *K-stable(semistable)* if $Fut(\mathcal{X}, \mathcal{L}) > 0(\geq 0)$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{L})$. Similarly, (X, D, L) is called *K-stable(semistable) with respect to angle β* if $Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) > 0(\geq 0)$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$.

When the central fiber $(\mathcal{X}_0, \mathcal{D}_0)$ is smooth, by Riemann-Roch the Futaki invariant then has a differential geometric expression as

$$Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) = \int_{\mathcal{X}_0} (S - \underline{S}) H \frac{\omega^n}{n!} - (1 - \beta) \left(\int_{\mathcal{D}_0} H \frac{\omega^{n-1}}{(n-1)!} - \frac{Vol(\mathcal{D}_0)}{Vol(\mathcal{X}_0)} \int_{\mathcal{X}_0} H \frac{\omega^n}{n!} \right),$$

where ω is an S^1 invariant Kähler metric in $2\pi c_1(\mathcal{L}_0)$ and H is the Hamiltonian function generating the S^1 action on \mathcal{L}_0 . This differs from the usual Futaki invariant by an extra term which reflects the cone angle.

The above abstract notion of K-stability is closely related to Chow stability for projective varieties, which we now recall. Given a \mathbb{C}^* action on \mathbb{CP}^N , and suppose the induced S^1 action preserves the Fubini-Study metric. Then the infinitesimal generator is given by a Hermitian matrix, say A . The Hamiltonian function for the S^1 action on \mathbb{CP}^N is

$$H_A(z) = \frac{z^* A z}{|z|^2}.$$

Given a projective manifold V in \mathbb{CP}^N , we define the *center of mass* of V

$$\mu(V) = \int_V \frac{z z^*}{|z|^2} d\mu_{FS} - \frac{Vol(V)}{N+1} Id \in \sqrt{-1} \mathfrak{su}(N+1),$$

viewing \mathbb{CP}^N as a co-adjoint orbit in $\mathfrak{su}(N+1)$. Define the *Chow weight* of V with respect to A to be

$$CH(V, A) = -Tr(\mu(V) \cdot A) = - \int_V H_A d\mu_{FS} + \frac{Vol(V)}{N+1} Tr A.$$

Notice this vanishes if A is a scalar matrix. The definition is not sensitive to singularities of V so one may define the Chow weight of any algebraic cycles

in a natural way. It is well-known that the $CH(e^{tA}.V, A)$ is a decreasing function of t , see for example [4]. So

$$CH(V, A) \leq CH(V_\infty, A), \quad (1)$$

where V_∞ is the limiting Chow cycle of $e^{tA}.V$ as $t \rightarrow -\infty$. V_∞ is fixed by the \mathbb{C}^* action and then $CH(V_\infty, A)$ is an algebraic geometric notion, i.e. independent of the Hermitian metric we choose on \mathbb{C}^{N+1} .

This well-known theory readily extends to pairs, see [5], [1]. We consider a pair of varieties (V, W) in \mathbb{CP}^N where W is a subvariety of V . Given a parameter $\lambda \in [0, 1]$, we define the *center of mass of (V, W) with parameter λ*

$$\mu(V, W, \lambda) = \lambda \int_V \frac{zz^*}{|z|^2} d\mu_{FS} + (1-\lambda) \int_W \frac{zz^*}{|z|^2} d\mu_{FS} - \frac{\lambda \text{Vol}(V) + (1-\lambda) \text{Vol}(W)}{N+1} Id,$$

and the *Chow weight* with parameter λ :

$$CH(V, W, A, \lambda) = -Tr(\mu(V, W, \lambda) \cdot A).$$

A pair (V, W) with vanishing center of mass with parameter λ is called a *λ -balanced embedding*.

Now given a test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$, it is explained in [13] and [4] (see also [12]) that for k large enough one can realize it by a family of projective schemes in $\mathbb{P}(H^0(X, L^k)^*)$ with a one parameter group action. Moreover one could arrange that the fiber $(\mathcal{X}_1, \mathcal{D}_1, \mathcal{L}_1)$ is embedded into $\mathbb{P}(H^0(X, L^k)^*)$ with a prescribed Hermitian metric, and the \mathbb{C}^* action is generated by a Hermitian matrix $-A_k$ (negative sign because we are taking the dual). Then as in [4] the Futaki invariant is the limit of Chow weight:

$$\lim_{k \rightarrow \infty} k^{-n} CH_k(\mathcal{X}_0, \mathcal{D}_0, -A_k, \lambda) = Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta), \quad (2)$$

with $\beta = \frac{3\lambda-2}{\lambda}$.

3 Proof of the main theorem

From now on we assume X is a Fano manifold of dimension n , D is a smooth anti-canonical divisor and the polarization is given by $L = -K_X$. We first prove the part of unstability in theorem 1.1, by considering the deformation to the normal cone of D , as studied by Ross-Thomas([14]). We blow up $D \times \{0\}$ in the total space $X \times \mathbb{C}$ and get a family $\pi : \mathcal{X} \rightarrow \mathbb{C}$. The exceptional divisor P is equal to the projective completion $\mathbb{P}(\nu_D \oplus \mathbb{C})$ of

the normal bundle ν_D in X . The central fiber \mathcal{X}_0 is the gluing of P to X along $D = \mathbb{P}(\nu_D)$. There is a \mathbb{C}^* action on \mathcal{X} coming from the trivial action on X and the standard \mathbb{C}^* action on \mathbb{C} . Let \mathcal{D} be the proper transform of $D \times \mathbb{C}$. This is \mathbb{C}^* invariant, and its intersection with the central fiber is the zero section $\mathbb{P}(\mathbb{C}) \subset \mathbb{P}(\nu_D \oplus \mathbb{C})$ (The readers are referred to [14] for a very nice picture of a deformation to the normal cone). The line bundle we use is $\mathcal{L}_c = L(-cP)$ (c is rational). It is shown in [14] that \mathcal{L}_c is ample when $c \in (0, 1)$. There is also a natural lift of the \mathbb{C}^* action to \mathcal{L}_c , so that we get test configurations $(\mathcal{X}, \mathcal{D}, \mathcal{L}_c)$ parametrized by c . We follow [14] to compute the Futaki invariant. We have the decomposition

$$H^0(\mathcal{X}, \mathcal{L}_c^k) = \bigoplus_{i=1}^{ck} t^{ck-i} H^0(X, L^{k-i}) \oplus t^{ck} \mathbb{C}[t] H^0(X, L^k).$$

Using the short exact sequence

$$0 \rightarrow H^0(X, L^{i-1}) \rightarrow H^0(X, L^i) \rightarrow H^0(D, L^i) \rightarrow 0,$$

we obtain

$$\begin{aligned} H^0(\mathcal{X}_0, \mathcal{L}_c^k) &= H^0(\mathcal{X}, \mathcal{L}_c^k) / t H^0(\mathcal{X}, \mathcal{L}_c^k) \\ &= H^0(X, L^{(1-c)k}) \oplus \bigoplus_{i=0}^{ck-i} t^{ck-i} H^0(D, L^{k-i}). \end{aligned}$$

This is indeed the weight decomposition of $H^0(\mathcal{X}_0, \mathcal{L}_c^k)$ under the \mathbb{C}^* action. Note the weight is -1 on t . So

$$\dim H^0(\mathcal{X}, \mathcal{L}_c^k) = \dim H^0(X, L^{(1-c)k}) + \sum_{i=0}^{ck-1} \dim H^0(D, L^{k-i}) = \dim H^0(X, L^{ck}).$$

This actually shows the flatness of the family $(\mathcal{X}, \mathcal{D}, \mathcal{L})$. Thus by Riemann-Roch,

$$a_0 = \frac{1}{n!} \int_X c_1(L)^n,$$

and

$$a_1 = \frac{1}{2(n-1)!} \int_X c_1(-K_X) \cdot c_1(L)^{n-1} = \frac{na_0}{2}.$$

The weight is given by

$$\begin{aligned} w_k &= - \sum_{i=0}^{ck-1} (ck-i) \dim H^0(D, L^{k-i}) \\ &= - \sum_{i=0}^{ck-1} (ck-i) \left(\frac{(k-i)^{n-1}}{(n-1)!} \int_D c_1(L)^{n-1} + O(k^{n-3}) \right) \\ &= -na_0 \int_0^c (c-x)(1-x)^{n-1} dx \cdot k^{n+1} - \frac{nca_0}{2} k^n + O(k^{n-1}). \end{aligned}$$

So

$$b_0 = \left(\frac{1 - (1 - c)^{n+1}}{n + 1} - c \right) a_0,$$

and

$$b_1 = -\frac{nc a_0}{2}.$$

Thus the ordinary Futaki invariant for the test configuration $(\mathcal{X}, \mathcal{L})$ is given by

$$Fut_c(\mathcal{X}, \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} = n \left(\frac{1 - (1 - c)^{n+1}}{n + 1} \right) a_0.$$

Note

$$H^0(\mathcal{D}, \mathcal{L}_c^k) = H^0(D \times \mathbb{C}, L^k \otimes (t)^{ck}) = t^{ck} \mathbb{C}[t] H^0(D, L^k).$$

So

$$H^0(\mathcal{D}_0, \mathcal{L}_c^k) = H^0(\mathcal{D}, \mathcal{L}_c^k) / t H^0(\mathcal{D}, \mathcal{L}_c^k) = t^{ck} H^0(D, L^k).$$

Thus we see

$$\tilde{a}_0 = \int_D \frac{c_1(L)^n}{(n-1)!} = n a_0,$$

and

$$\tilde{b}_0 = -c \int_D \frac{c_1(L)^n}{(n-1)!} = -n c a_0.$$

Therefore,

$$\begin{aligned} Fut_c(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) &= Fut_c(\mathcal{X}, \mathcal{L}) + (1 - \beta) \left(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0} b_0 \right) \\ &= \left[n \left(\frac{1 - (1 - c)^{n+1}}{n + 1} \right) + (1 - \beta) \left(-n c + n \left(c - \frac{1 - (1 - c)^{n+1}}{n + 1} \right) \right) \right] a_0 \\ &= n \beta \frac{1 - (1 - c)^{n+1}}{n + 1} a_0. \end{aligned}$$

Therefore for $\beta < 0$ this particular test configuration gives rise to instability, and for $\beta = 0$ the pair (X, D) can not be stable.

Now we move on to prove K-semistability for $\beta = 0$. Using again the short exact sequence

$$0 \rightarrow H^0(X, L^{j-1}) \rightarrow H^0(X, L^j) \rightarrow H^0(D, L^j) \rightarrow 0$$

successively we can choose a splitting

$$H^0(X, L^k) = H^0(X, L^{s-1}) \oplus \bigoplus_{j=s}^k H^0(D, L^j) \quad (3)$$

for s large enough and all $k > s$. By Yau's theorem([19]) there is a unique Ricci flat metric ω_0 in $c_1(L)|_D$. This defines a Hermtian metric on $H^0(D, L^j)$

by the L^2 inner product. We can put an arbitrary metric on $H^0(X, L^{s-1})$, and make the splitting (3) orthogonal. We also identify the vector spaces with their duals using these metrics. Take s large enough so that D embeds into $\mathbb{P}(H^0(D, L^j))$ and X embeds into $\mathbb{P}(H^0(X, L^j))$ for all $j \geq s-1$. Choosing an orthonormal basis of $H^0(D, L^j)$ we get an embedding $f_j : D \rightarrow \mathbb{P}(H^0(D, L^j)) \cong \mathbb{P}^{n_j-1}$ (Here $n_j = \dim H^0(D, L^j)$). We also pick an arbitrary embedding $f_{s-1} : X \rightarrow \mathbb{P}(H^0(X, L^{s-1}))$. Denote by D_j the image of f_j , and let $N(D_{j-1}, D_j)$ be the variety consisting of all points in $\mathbb{P}(H^0(D, L^{j-1}) \oplus H^0(D, L^j)) \subset \mathbb{P}(H^0(X, L^k))$ of the form $[uf_{j-1}(p) : vf_j(p)]$ for $p \in D$ and $u, v \in \mathbb{C}$. The projection map $\pi_j : N(D_{j-1}, D_j) \rightarrow D$ makes it a \mathbb{P}^1 bundle over D . This is isomorphic to the projective completion of the normal bundle of D in X . Let X_k be the union of all these $N(D_{j-1}, D_j)$ ($s \leq j \leq k$) together with $f_{s-1}(X)$. Then it is not hard to see that as a pair of Chow cycles (X_k, D_k) lies in the closure of the $PGL(d_k; \mathbb{C})$ orbit of a smooth embedding of (X, D) in $\mathbb{P}(H^0(X, L^k))$. We want to estimate its center of mass. The following two lemmas involve some calculation and the proof will be deferred to the end of this section.

Lemma 3.1. *For $s \leq j \leq k$ we have*

$$\pi_{j*} \omega_{FS}^n = \sum_{i=0}^{n-1} \omega_j^i \omega_{j-1}^{n-1-i},$$

where $\omega_j = f_j^* \omega_{FS}$.

This lemma implies that

$$Vol(N(D_{j-1}, D_j)) = \frac{1}{n!} \sum_{i=0}^{n-1} j^i (j-1)^{n-1-i} \cdot (n-1)! Vol(D) = (j^n - (j-1)^n) Vol(X).$$

Summing over j we see that $Vol(X_k) = k^n Vol(X)$.

Notice $N(D_{j-1}, D_j)$ can only contribute to the $H^0(D, L^{j-1})$ and $H^0(D, L^j)$ components of the center of mass of X_k . Denote by $Z_j = (Z_j^1, \dots, Z_j^{n_j})$ the homogeneous coordinates on $H^0(D, L^j)$ for $s \leq j \leq k$, and by Z_{s-1} the homogeneous coordinate on $H^0(X, L^{s-1})$. Then we have

Lemma 3.2. *For $s \leq j \leq k$ we have*

$$\begin{aligned} \pi_{j*} \frac{Z_j Z_{j-1}^*}{|Z_j|^2 + |Z_{j-1}|^2} \omega_{FS}^n &= 0, \\ \pi_{j*} \frac{Z_{j-1} Z_j^*}{|Z_j|^2 + |Z_{j-1}|^2} \omega_{FS}^n &= 0, \\ \pi_{j*} \frac{Z_j Z_j^*}{|Z_j|^2 + |Z_{j-1}|^2} \omega_{FS}^n &= \frac{Z_j Z_j^*}{|Z_j|^2} \cdot \sum_{i=0}^{n-1} \frac{i+1}{n+1} \omega_j^i \omega_{j-1}^{n-1-i}, \end{aligned}$$

$$\pi_{j^*} \frac{Z_{j-1} Z_{j-1}^*}{|Z_j|^2 + |Z_{j-1}|^2} \omega_{FS}^n = \frac{Z_{j-1} Z_{j-1}^*}{|Z_{j-1}|^2} \cdot \sum_{i=0}^{n-1} \frac{n-i}{n+1} \omega_j^i \omega_{j-1}^{n-1-i},$$

This lemma implies that the center of mass $\mu(X_k)$ also splits as the direct sum of μ_j 's. For j between s and $k-1$ we have

$$\mu_j(X_k) = \int_{X_k} \frac{Z_j Z_j^*}{|Z|^2} \frac{\omega_{FS}^n}{n!} = \frac{1}{n!} \int_D \frac{Z_j Z_j^*}{|Z_j|^2} \sum_{i=0}^{n-1} \left(\frac{i+1}{n+1} \omega_j^i \omega_{j-1}^{n-1-i} + \frac{n-i}{n+1} \omega_{j+1}^i \omega_j^{n-1-i} \right),$$

while

$$\mu_k(X_k) = \int_{X_k} \frac{Z_k Z_k^*}{|Z|} \frac{1}{n!} \omega_{FS}^n = \frac{1}{n!} \int_D \frac{Z_k Z_k^*}{|Z_k|^2} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \omega_k^i \omega_{k-1}^{n-1-i},$$

and

$$\mu_{s-1}(X_k) = \frac{1}{n!} \int_D \frac{Z_{s-1} Z_{s-1}^*}{|Z_{s-1}|^2} \sum_{i=0}^{n-1} \frac{n-i}{n+1} \omega_s^i \omega_{s-1}^{n-1-i} + \int_{X_{s-1}} \frac{Z_{s-1} Z_{s-1}^*}{|Z_{s-1}|^2} \frac{\omega_{FS}^n}{n!}.$$

The induced metric ω_j is related to the original metric ω_0 by the “density of state” function:

$$\omega_j = j\omega_0 + \sqrt{-1} \partial \bar{\partial} \log \rho_j(\omega_0).$$

It is well-known that we have the following expansion (see [2], [21], [8], [10])

$$\rho_j(\omega_0) = j^{n-1} + \frac{S(\omega_0)}{2} j^{n-2} + O(j^{n-3}) = j^{n-1} + O(j^{n-3}),$$

since ω_0 is Ricci flat. Thus

$$\omega_j^i \omega_{j-1}^{n-1-i} = j^i (j-1)^{n-1-i} \omega_0^{n-1} (1 + O(j^{-3})).$$

To estimate μ_j recall we have chosen an orthonormal basis $\{s_j^l\}$ of $H^0(D, L^j)$ and we can assume μ_j is a diagonal matrix. Then for $s \leq j \leq k-1$ we obtain

$$\mu_j^l(X_k) = \int_D \frac{|s_j^l|^2 (1 + O(j^{-3}))}{j^{n-1} + O(j^{n-3})} \sum_{i=0}^{n-1} \left(\frac{i+1}{n+1} j^i (j-1)^{n-1-i} + \frac{n-i}{n+1} (j+1)^i j^{n-1-i} \right) \frac{\omega_0^{n-1}}{n!}.$$

It is easy to see that

$$\sum_{i=0}^{n-1} \left(\frac{i+1}{n+1} j^i (j-1)^{n-1-i} + \frac{n-i}{n+1} (j+1)^i j^{n-1-i} \right) = nj^{n-1} + O(j^{n-3}).$$

Thus

$$\mu_j^l(X_k) = 1 + O(j^{-2}).$$

For $j = k$, we have

$$\mu_k^l(X_k) = 1/2 + O(k^{-1}).$$

For $j = s - 1$, we have

$$\mu_{s-1}^l(X_k) = O(1).$$

The center of mass of the pair (X_k, D_k) with respect to $\lambda = 2/3$ is given by

$$\mu(X_k, D_k, 2/3) = \frac{2}{3}\mu(X_k) + \frac{1}{3}\mu(D_k) - \underline{\mu} \cdot Id,$$

where we denote

$$\underline{\mu} = \frac{2Vol(X_k) + Vol(D_k)}{3d_k} = \frac{2}{3} + O(k^{-2}).$$

Thus for $s \leq j \leq k - 1$ and $0 \leq l \leq n_j$ we have

$$\mu_j^l(X_k, D_k, 2/3) = O(j^{-2}) + O(k^{-2}).$$

Since n_j is a polynomial of degree $n - 1$ in j , we obtain

$$|\mu_j(X_k, D_k, 2/3)|_2 = \left(\sum_{l=0}^{n_j} |\mu_j^l(X_k, D_k, 2/3)|^2 \right)^{1/2} = O(j^{\frac{n-5}{2}}),$$

and

$$\sum_{j=s}^{k-1} |\mu_j(X_k, D_k, 2/3)|_2 = O(k^{\frac{n-3}{2}}).$$

For $j = k$, we have

$$\mu_k^l(D_k) = \int_D \frac{|s_k^l|^2}{k^{n-1} + O(k^{n-3})} (1 + O(k^{-2})) \frac{k^{n-1} \omega_0^{n-1}}{(n-1)!} = 1 + O(k^{-2}).$$

So

$$\mu_k^l(X_k, D_k) = O(k^{-1}),$$

and

$$|\mu_k(X_k, D_k)|_2 = O(k^{\frac{n-3}{2}}).$$

Therefore we obtain

$$|\mu(X_k, D_k)|_2 = O(k^{\frac{n-3}{2}}).$$

So for a smoothly embedded (X, D) in $\mathbb{P}(H^0(X, L^k))$ we have

$$\inf_{g \in PGL(d_k; \mathbb{C})} |\mu(g \cdot (X, D))|_2 = O(k^{\frac{n-3}{2}}).$$

In particular there are embeddings $\iota_k : (X, D) \rightarrow \mathbb{P}(H^0(X, L^k))$ such that

$$|\mu(\iota_k(X, D))|_2 = O(k^{\frac{n-3}{2}}).$$

Now any test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ can be represented by a family in $\mathbb{P}(H^0(X, L^k))$ such that the fiber $(\mathcal{X}_1, \mathcal{D}_1, \mathcal{L}_1)$ is embedded by ι_k and the \mathbb{C}^* action is generated by a Hermitian matrix A_k . Again by general theory $|A_k|_2^2 = \text{Tr} A_k^2 = O(k^{n+2})$. Therefore by monotonicity of the Chow weight we obtain

$$\begin{aligned} CH_k(\mathcal{X}_0, \mathcal{D}_0, -A_k, 2/3) &\geq CH_k(\mathcal{X}_1, \mathcal{D}_1, -A_k, 2/3) \\ &\geq - \inf_{g \in PGL(d_k; \mathbb{C})} |\mu(g \cdot (X, D))|_2 \cdot |-A_k|_2 \\ &\geq -O(k^{n-\frac{1}{2}}). \end{aligned}$$

Thus by (2)

$$Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, 0) = \lim_{k \rightarrow \infty} k^{-n} CH_k(\mathcal{X}_0, \mathcal{D}_0, -A_k, \frac{2}{3}) \geq 0.$$

This finishes the proof of theorem 1.1.

Now we prove the lemma 3.1 and 3.2. In general suppose there are two embeddings $f_1 : D \rightarrow \mathbb{P}^l$ and $f_2 : D \rightarrow \mathbb{P}^m$. As before, let $N(D)$ be the variety in \mathbb{P}^{l+m+1} containing all points of the form $(tf_1(x), sf_2(x))$ where $t, s \in \mathbb{C}$. Intuitively $N(D)$ is ruled by all lines connecting $f_1(x)$ and $f_2(x)$ for $x \in D$. Choose a local coordinate chart U in D such that the image $f_1(U)$ and $f_2(U)$ are contained in a standard coordinate chart for the projective spaces \mathbb{P}^l and \mathbb{P}^m respectively. Let $[1 : z]$ and $[1 : w]$ be local coordinates in \mathbb{P}^l and \mathbb{P}^m . Under unitary transformations we may assume $f_1(x_0) = [1 : 0]$ and $f_2(x_0) = [1 : 0]$. The line connecting $f_1(x_0)$ and $f_2(x_0)$ is parametrized as $[1 : 0 : t : 0]$ for $t \in \mathbb{C}$. Along this line we have

$$\begin{aligned} \omega_{FS} &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + |z|^2 + |t|^2 + |t|^2 |w|^2) \\ &= \frac{\sqrt{-1}}{2\pi} \cdot \frac{(1 + |t|^2) \sum_i dz^i d\bar{z}^i + |t|^2(1 + |t|^2) \sum_j dw^j d\bar{w}^j + dt d\bar{t}}{(1 + |t|^2)^2}. \end{aligned}$$

Thus

$$\omega_{FS}^n = n \left(\frac{\sqrt{-1}}{2\pi} \right)^n (1 + |t|^2)^{-n-1} \left(\sum_i dz^i d\bar{z}^i + |t|^2 \sum_j dw^j d\bar{w}^j \right)^{n-1} dt d\bar{t}.$$

Hence integrating along the \mathbb{P}^1 we get

$$\begin{aligned} \int_{\mathbb{P}^1} \omega_{FS}^n &= \frac{1}{2\pi} \int_{\mathbb{C}} n(\omega_1 + |t|^2 \omega_2)^{n-1} (1 + |t|^2)^{-n-1} \sqrt{-1} dt d\bar{t} \\ &= \frac{1}{2\pi} \int_0^\infty n \sum_{j=0}^{n-1} \binom{n-1}{j} \omega_1^j \omega_2^{n-1-j} x^j (1+x)^{-n-1} dx \\ &= \sum_{j=0}^{n-1} \omega_1^j \omega_2^{n-1-j}. \end{aligned}$$

This proves lemma 3.1.

For the center of mass we compute

$$\int_{\mathbb{P}^1} \frac{1}{1+|t|^2} \omega_{FS}^n = \sum_{j=0}^{n-1} \frac{j+1}{n+1} \omega_1^j \omega_2^{n-1-j},$$

and

$$\int_{\mathbb{P}^1} \frac{|t|^2}{1+|t|^2} \omega_{FS}^n = \sum_{j=0}^{n-1} \frac{n-j}{n+1} \omega_1^j \omega_2^{n-1-j}.$$

Thus globally we obtain

$$\int_{N(D)} \frac{zz^*}{|z|^2 + |w|^2} \omega_{FS}^n = \int_D \frac{zz^*}{|z|^2} \sum_{j=0}^{n-1} \frac{j+1}{n+1} \omega_1^j \omega_2^{n-1-j},$$

and

$$\int_{N(D)} \frac{ww^*}{|z|^2 + |w|^2} \omega_{FS}^n = \int_D \frac{ww^*}{|w|^2} \sum_{j=0}^{n-1} \frac{n-j}{n+1} \omega_1^j \omega_2^{n-1-j}.$$

Also notice by symmetry of $N(D)$ under the map $w \mapsto -w$ we have

$$\int_{N(D)} \frac{zw^*}{|z|^2 + |w|^2} \omega_{FS}^n = 0.$$

Similarly

$$\int_{N(D)} \frac{wz^*}{|z|^2 + |w|^2} \omega_{FS}^n = 0.$$

This proves lemma 3.2.

Remark 3.3. In the case when X is \mathbb{P}^1 and D consists of two points, one can indeed find the precise balanced embedding for $\lambda = 2/3$. In \mathbb{P}^k let L be the chain of lines L_i connecting p_i and p_{i+1} ($0 \leq i \leq k-1$), where p_i is the i -th coordinate point. Then it is easy to see that L is the degeneration limit of a smooth degree k rational curve, and it is exactly $\frac{2}{3}$ -balanced. It is well-known that a rational normal curve in \mathbb{P}^k is always Chow polystable, it follows by linearity that it is also Chow polystable for $\lambda \in (2/3, 1]$.

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